THEORY OF SEGMENTED ELECTRODIFFUSION PROBES: THE EFFECT OF INSULATING INSERTIONS

Ondřej WEIN and Kamil WICHTERLE

Institute of Chemical Process Fundamentals,
Czechoslovak Academy of Sciences, 165 02 Prague 6-Suchdol

Received February 9, 1989
Accepted April 27, 1989

Simple formulae are given for the calculation of limiting diffusion currents to a series of three strip electrodes separated by insulating insertions.

The direction-sensitive electrodiffusion probes consist of several working electrodes (segments) which are separated from one another by thin insulating insertions[1-3]. It has been assumed in the basic theory[4] of multisegmented probes that these insertions are thin enough to preserve the same current distribution as for simple connected probes. Recent calibration experiments[5] have shown a remarkable effect of the insertions on the directional characteristics.

The theory of convective diffusion to a smooth surface with both the active and inactive parts has been developed in several papers[6-10]. Unfortunately, the results were expressed merely in the form of multidimensional functional expansions or Lebesgue integrals, and therefore they cannot be effectively used. The purpose of the present paper is to analyse theoretically the current distribution for three-segment probes with insulating insertions and to represent it in a form suitable for numerical computations.

THEORETICAL

Statement of the Problem

We consider the convective diffusion of a depolarizer from the bulk of a streaming liquid to a planar solid surface. The velocity field is unidirectional with a constant shear rate q,

$$v_x = qz\,, \quad v_y = v_z = 0\,. \tag{1}$$

The bulk concentration of the depolarizer is constant,

$$c \to c_0\,, \quad \text{for} \quad z \to \infty\,. \tag{2}$$

The solid surface, located in the plane $z = 0$, consists of a series of parallel strips which are either perfectly active,

$$c = 0\,, \quad \text{for} \quad z = 0\,, \tag{3}$$

or perfectly inactive,

$$\mathrm{d}_z c = 0\,, \quad \text{for} \quad z = 0\,. \tag{4}$$

The transport equation is considered in the common form,

$$qz\, \mathrm{d}_x c = D\, \mathrm{d}^2_{zz} c\,, \tag{5}$$

which neglects the effect of longitudinal diffusion.

For a redox reaction with the transfer of n electrons per one molecule of the depolarizer, the current density J is given by the relation

$$J = nFD\left(-\mathrm{d}_z c|_{z=0}\right). \tag{6}$$

The current to a part of the surface can then be calculated by integrating the expression (6) over the surface. We limit ourselves to planar symmetry. The probe consists of a system of parallel strips of the same width w. The forward edge of the first active strip is placed at $x = 0$. Under these specifications, the current to the surface between the lines $x = 0$ and $x = \text{const} > 0$ is given by the integral

$$I(x) = wnFD \int_0^x \left(-\mathrm{d}_z c|_{z=0}\right) \mathrm{d}x\,. \tag{7}$$

If the k-th strip begins at $x = x_{2k-2}$ and ends at $x = x_{2k-1}$, the partial current to this strip, I_k, $k = 1, 2, \ldots$, see also Fig. 1., is given by the difference

$$I_k = I(x_{2k-1}) - I(x_{2k-2})\,. \tag{8}$$

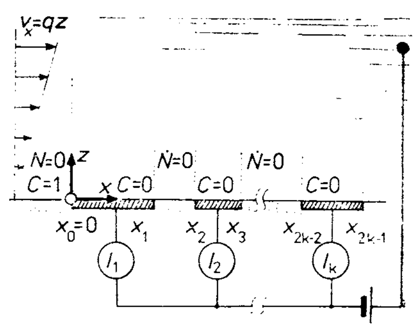

FIG. 1

Transport configuration of a series of strip segments separated by insulating insertions; $C = 0$ active segments, $\dot{N} = 0$ insulating insertions

For convenience, we introduce the normalized concentration field and the related variables C^*, Z, and X:

$$C^* = 1 - c/c_0\,, \quad Z = z/\mu\,, \quad X = x/\mu\,, \tag{9}$$

$$\mu = \sqrt{(9D/q)}\,. \tag{10}$$

A special notation is introduced for the normalized diffusion current and concentration at the boundary:

$$C(X) = c(z, x)|_{z=0}/c_0\,, \tag{11}$$

$$N(X) = \Gamma(\tfrac{4}{3})\, I(x)/(wnFc_0D)\,. \tag{12}$$

The boundary value problem for an unknown concentration field C^* can now be written in the following form:

$$\begin{aligned} 9Z\, \mathrm{d}_X C^* &= \mathrm{d}^2_{ZZ} C^*\,, \quad \text{for} \quad X > 0, Z > 0\,, \\ C^*(Z, X) &= 0\,, \quad \text{for} \quad X < 0\,, \\ C^*(Z, X) &\to 0\,, \quad \text{for} \quad Z \to \infty\,. \end{aligned} \tag{13}$$

The boundary conditions on the solid surface, $Z = 0$, specify either a concentration distribution,

$$C^*(Z, X)|_{Z=0} = 1 - C(X)\,, \quad \text{for} \quad X > 0\,, \tag{14}$$

or a distribution of current density,

$$\mathrm{d}_Z C^*(Z, X)|_{Z=0} = \dot{N}(X)/\Gamma(\tfrac{4}{3})\,, \quad \text{for} \quad X > 0\,, \tag{15}$$

where the dot stands for the longitudinal derivative, $\dot{N} = \mathrm{d}N/\mathrm{d}X$.

If one of the influence functions $C(X)$, $N(X)$ is given, the problem is fully stated (well-posed) and can be solved by using an appropriate mathematical technique. In particular, the functions C and N are unambiguously related to each other by the stated boundary value problem.

Starting Integral Equations

It is the decisive point of the analysis to find an explicit functional relation between the functions C and N. Generally speaking, such relations are well known for the linear parabolic system under consideration[11]. However, it may be useful to refresh briefly a path to the resulting integral equations.

We shall make use of the Laplace transform. In general, the L-transform of a function $\mathrm{f}(X)$, $X \geqq 0$, is given as

$$\bar{\mathrm{f}}(p) = \mathrm{L}\{\mathrm{f}(X)\} \equiv \int_0^\infty \exp(-pX)\,\mathrm{f}(X)\,\mathrm{d}X\,. \tag{16}$$

In particular:

$$\bar{C}(Z, p) = \mathrm{L}\{C^*(Z, X)\}\,, \tag{17}$$

$$\mathrm{d}_Z\bar{C}(Z, p) = \mathrm{L}\{\mathrm{d}_Z C^*(Z, X)\}\,, \tag{18}$$

$$p\bar{C}(Z, p) - C^*(Z, 0_+) = \mathrm{L}\{\mathrm{d}_X C^*(Z, X)\}\,, \tag{19}$$

and

$$p\bar{N}(p) - N(0_+) = \mathrm{L}\{\dot{N}(X)\}\,, \tag{20}$$

$$\bar{C}^w(p) \equiv \mathrm{L}\{C^*(0, X)\}\,, \tag{21}$$

$$\bar{N}(p) = \mathrm{L}\{N(X)\} = -\Gamma(\tfrac{4}{3})\,p^{-1}\,\mathrm{d}_Z\bar{C}(Z, p)|_{Z=0}\,, \tag{22}$$

where $N(0_+) = 0$.

The problem for a prescribed surface concentration has the following representation in the L-domain:

$$9Zp\bar{C} = \mathrm{d}_{ZZ}^2\bar{C}\,, \tag{23}$$

$$\bar{C}(Z, p)|_{Z=0} = \bar{C}^w(p)\,, \tag{24}$$

$$\bar{C}(Z, p)|_{Z=\infty} = 0\,. \tag{25}$$

This is an ordinary boundary-value problem with the single independent variable, $Z \in (0, \infty)$, containing two parameters p, $\bar{C}^w$. Because of the linearity of the problem, knowledge of any non-trivial solution $\bar{C}_0(Z, p)$ is sufficient to express the general one:

$$\bar{C}(Z, p) = \bar{C}_0(Z, p)\,(\bar{C}^w(p)/\bar{C}_0^w(p))\,. \tag{26}$$

As a consequence, the corresponding fluxes are given as:

$$\bar{N}(p) = \bar{N}_0(p)\,(\bar{C}^w(p)/\bar{C}_0^w(p))\,, \tag{27}$$

where

$$\bar{N}_0(p) = -\Gamma(\tfrac{4}{3})\,p^{-1}\,\mathrm{d}_Z\bar{C}_0(Z, p)|_{Z=0}\,. \tag{28}$$

We actually know certain explicit solutions to the problem. At constant surface concentration, the solution has been given by Leveque:

$$C^*(Z, X) = \frac{1}{\Gamma(\frac{1}{3})}\int_{Z^3/X}^{\infty} \exp(-s)\,s^{-2/3}\,\mathrm{d}s\,. \tag{29}$$

In particular:

$$C(X) = 0\,, \quad N(X) = \tfrac{3}{2}X^{2/3}\,, \tag{30}$$

$$\bar{C}^{w}(p) = p^{-1}\,, \quad \bar{N}(p) = \Gamma(\tfrac{2}{3})\,p^{-5/3}\,. \tag{31}$$

Another simple explicit solution is known at constant current density:

$$C^{*}(Z, X) = \frac{3}{B}\left[X^{1/3}\exp\left(-Z^{3}/X\right) - Z\int_{Z^{3}/X}^{\infty}\exp\left(-s\right)s^{-1/3}\,\mathrm{d}s\right]. \tag{32}$$

In particular:

$$N(X) = X\,, \quad C(X) = 1 - 3X^{1/3}/B\,, \tag{33}$$

$$\bar{N}(p) = p^{-2}\,, \quad \bar{C}^{w}(p) = p^{-4/3}/\Gamma(\tfrac{4}{3})\,. \tag{34}$$

The both particular solutions result in the same relation:

$$\bar{N}(p)/\bar{C}^{w}(p) = \Gamma(\tfrac{4}{3})\,p^{-2/3}\,, \tag{35}$$

which is valid for any admissible longitudinal courses of surface concentrations or diffusion currents.

By applying the convolution theorem[11], various integral relations can be derived between the original influence functions C and N. The following two representations appear to be most useful in solving the problem under consideration:

$$N(X) = \textstyle\int_{0}^{X}(X - t)^{-1/3}\left[1 - C(t)\right]\mathrm{d}t\,, \tag{36}$$

$$C(X) = \frac{1}{B}\int_{0}^{X}(X - s)^{-2/3}\left[s^{-1/3} - \dot{N}(s)\right]\mathrm{d}s\,. \tag{37}$$

Current Distributions for a Series of Strip Segments

The considered transport configuration has been shown in Fig. 1, where

$$X_{k} = x_{k}/\mu\,. \tag{38}$$

The probe consists of several parallel strip segments separated by strip insulating insertions. Our aim is to determine the normalized current distribution $N(X)$ by solving the integral equations (*36*) and (*37*) under the following conditions:

$$N(X) = 0\,, \quad C(X) = 1\,, \quad \text{for} \qquad X < 0\,, \tag{39a}$$

$$C(X) = 0\,, \quad \text{for} \quad X_{2k-2} < X < X_{2k-1}\,, \tag{39b}$$

$$\dot{N}(X) = 0\,, \quad \text{for} \quad X_{2k-1} < X < X_{2k}\,. \tag{39c}$$

From the integral equations (*36*) and (*37*) it is apparent that the fluxes or concentrations can be calculated to the extent in which the counterpart distributions are known in advance. Because the functions $C(X)$ and $N(X)$ are alternately known from the conditions (*39a, b, c*), the simultaneous application of the integral equations (*36*) and (*37*) results in an obvious recursive algorithm. Our main task is the conversion of the resulting multidimensional integrals to more appropriate computational formulae. The technical details of the derivation are given in the Appendix which should be consulted, e.g., for the definitions of the auxiliary functions F and Φ and the constant B.

Flux to the first segment, $X \in (0, X_1)$. After the starting concentration jump at the point $X = X_0 = 0$, the concentration at the surface of the first segment is zero. The corresponding current distribution, identical to the well-known Leveque result, can be obtained from Eq. (*36*):

$$N(X) = \tfrac{3}{2} X^{2/3}\,. \tag{40}$$

Concentration at the first insertion, $X \in (X_1, X_2)$. The calculation goes still straightforwardly by direct using of Eq. (*37*):

$$C_1(X) = \frac{1}{B}\left[\int_0^{X_1} + \int_{X_1}^{X}\right](X - s)^{-2/3}\,[s^{-1/3} - \dot{N}(s)]\,.\,\mathrm{d}s =$$

$$= \frac{1}{B}\int_{X_1}^{X}(X - s)^{-2/3}\,s^{-1/3}\,\mathrm{d}s = [1 - \mathrm{F}(X_1/X)]\,. \tag{41}$$

Flux to the second segment, $X \in (X_2, X_3)$. By applying Eq. (*36*), we obtain

$$N(X) = \tfrac{3}{2}X^{2/3} - \textstyle\int_{X_1}^{X_2}(X - t)^{-1/3}\,C_1(t)\,\mathrm{d}t =$$

$$= \tfrac{3}{2}X^{2/3} - \textstyle\int_{X_1}^{X_2}(X - t)^{-1/3}\,[1 - \mathrm{F}(X_1/t)]\,\mathrm{d}t\,, \tag{42}$$

which can be simplified to the following form:

$$N(X) = \tfrac{3}{2}X_1^{2/3} + \tfrac{3}{2}X^{2/3}\,\Phi(X_1/X, X_2/X)\,. \tag{43}$$

Concentration at the second insertion, $X \in (X_3, X_4)$. There are four local branches of $N(X)$ which must be known for direct application of Eq. (*37*):

$$s^{-1/3} - \dot{N}(s) = \begin{cases} 0, & 0 < s < X_1, \\ s^{-1/3}, & X_1 < s < X_2, \\ -\dfrac{1}{3}\displaystyle\int_{X_1}^{X_2} \frac{C_1(u)\,\mathrm{d}u}{(s-u)^{4/3}}, & X_2 < s < X_3, \\ s^{-1/3}, & X_3 < s < X_4. \end{cases} \tag{44}$$

The concentration distribution can then be expressed by means of two-dimensional integrals of the incomplete Beta function:

$$C_2(X) = \frac{1}{B}\left[\int_{X_1}^{X_2} + \int_{X_3}^{X}\right](X - s)^{-2/3}\, s^{-1/3}\,\mathrm{d}s -$$

$$- \frac{1}{3B}\int_{X_2}^{X_3}(X - s)^{-2/3}\,\mathrm{d}s\int_{X_1}^{X_2}(s - u)^{-4/3}\,[1 - \mathrm{F}(X_1/u)]\,\mathrm{d}u\,. \tag{45}$$

Flux to the third segment, $X \in (X_4, X_5)$. The starting representation,

$$N(X) = \tfrac{3}{2}X^{2/3} - \int_{X_1}^{X_2}\frac{C_1(t)\,\mathrm{d}t}{(X - t)^{1/3}} - \int_{X_3}^{X_4}\frac{C_2(t)\,\mathrm{d}t}{(X - t)^{1/3}}, \tag{46}$$

can be decomposed by using the known expressions for C_1 and C_2:

$$N_3(X) = \tfrac{3}{2}X^{2/3}(1 - Q_4 - Q_1 - Q_2 + Q_3), \tag{47}$$

where

$$\tfrac{3}{2}X^{2/3}Q_4 = \int_{X_3}^{X_4}\frac{1 - \mathrm{F}(X_3/t)}{(X - t)^{1/3}}\,\mathrm{d}t\,, \tag{48}$$

$$\tfrac{3}{2}X^{2/3}Q_1 = \int_{X_1}^{X_2}\frac{1 - \mathrm{F}(X_1/t)}{(X - t)^{1/3}}\,\mathrm{d}t\,, \tag{49}$$

$$\tfrac{3}{2}X^{2/3}Q_2 = \frac{1}{B}\int_{X_3}^{X_4}\frac{\mathrm{d}t}{(X - t)^{1/3}}\int_{X_1}^{X_2}\frac{\mathrm{d}s}{(t - s)^{2/3}\, s^{1/3}}, \tag{50}$$

$$\tfrac{3}{2}X^{2/3}Q_3 = \frac{1}{3B}\int_{X_3}^{X_4}\frac{\mathrm{d}t}{(X - t)^{1/3}}\int_{X_2}^{X_3}\frac{\mathrm{d}s}{(t - s)^{2/3}}\int_{X_1}^{X_2}\frac{1 - \mathrm{F}(X_1/u)}{(s - u)^{4/3}}\,\mathrm{d}u\,. \tag{51}$$

These terms can be reduced to quadratures and elementary functions:

$$Q_4 = 1 - \varkappa_3^{2/3} - \Phi(\varkappa_3, \varkappa_4)\,, \tag{52}$$

$$Q_1 = 1 - \varkappa_1^{2/3} - \Phi(\varkappa_1, \varkappa_2)\,, \tag{53}$$

$$Q_2 = \Phi(\varkappa_2, \varkappa_4) - \Phi(\varkappa_1, \varkappa_4) + \Phi(\varkappa_1, \varkappa_3) - \Phi(\varkappa_2, \varkappa_3)\,, \tag{54}$$

$$Q_3 = \frac{2}{3}\int_{\varkappa_2}^{\varkappa_3}\left[\mathrm{F}\left(\frac{1-\varkappa_3}{1-s}\right) - \mathrm{F}\left(\frac{1-\varkappa_4}{1-s}\right)\right]\left[\frac{1-\mathrm{F}\left(\frac{\varkappa_1}{\varkappa_2}\right)}{(s-\varkappa_2)^{1/3}} - \frac{1-\mathrm{F}\left(\frac{\varkappa_1}{\varkappa_2}\frac{s-\varkappa_2}{s-\varkappa_1}\right)}{s^{1/3}}\right]\mathrm{d}s\,, \tag{55}$$

where

$$\varkappa_k = X_k/X = x_k/x\,. \tag{56}$$

For the particular case where $X = X_5$, the term $N(X_5)$ gives the normalized overall current to the three-segment probe, and the quantities $\varkappa_k$ become the geometrical simplexes:

$$\varkappa_k = x_k/x_5\,, \quad (k = 1, 2, 3, 4)\,. \tag{57}$$

RESULTS

Partial Currents to Individual Segments

Eqs (*40*), (*43*), and (*47*) can be rearranged to give the following expressions for the partial currents I_1, I_2, I_3 to the individual (strip) segments of a three-segment probe:

$$I_1 = Kx_1^{2/3}\,, \tag{58}$$

$$I_2 = Kx_3^{2/3}\,\Phi\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right), \tag{59}$$

$$I_3 = Kx_5^{2/3}Q_5\left(\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5}\right), \tag{60}$$

where

$$K = \frac{3^{1/3}}{2\Gamma(\frac{4}{3})}\,wnFc_0D^{2/3}q^{1/3}\,, \tag{61}$$

$$\begin{aligned} Q_5(\varkappa_1, \varkappa_2, \varkappa_3, \varkappa_4) &= \Phi(\varkappa_3, \varkappa_4) - 1 + \Phi(\varkappa_1, \varkappa_2) - \\ &- \Phi(\varkappa_2, \varkappa_4) + \Phi(\varkappa_1, \varkappa_4) - \Phi(\varkappa_1, \varkappa_3) + \Phi(\varkappa_2, \varkappa_3) + \\ &+ \varkappa_3^{2/3}\left[1 - \Phi\left(\frac{\varkappa_1}{\varkappa_3}, \frac{\varkappa_2}{\varkappa_3}\right)\right] + Q_3\,. \end{aligned} \tag{62}$$

Asymptotic Representations of Partial Currents

There are two extremal transport situations with apparent physical meaning – the case of asymptotically large or small insertions, as compared with the lengths of the segments.

For the case of large insertions, every segment behaves as a single electrode and the partial currents are given by the Leveque formula, see Eq. (*40*). The expression (*59*) can be, for $x_2 - x_1 \gg x_3 - x_2$, $x_2 - x_1 \gg x_1$, approximated in the following way by using the asymptotic expansions (*A22*), (*A24*):

$$I_2 \approx K(x_3 - x_2)^{2/3}\left[1 - \frac{3}{2B}\left[\frac{x_1}{x_3}\right]^{2/3} + \ldots\right], \tag{63}$$

where the first term represents the Leveque formula.

In a tedious but straightforward way it is possible to obtain an analogous result for the third segment:

$$I_3 \approx K(x_5 - x_4)^{2/3}\left[1 - \frac{3}{2B}\left[\left[\frac{x_1}{x_5}\right]^{2/3} + \left[\frac{x_3 - x_2}{x_5}\right]^{2/3}\right] + \ldots\right]. \tag{64}$$

For small insertions, the current distribution over the entire three-segmented probe is nearly the same as for a single electrode, and hence the Leveque formula can be used for calculation of the total current between the forward edge of the probe and the boundaries of the individual segments. The expression (*59*) can be, for $x_2 - x_1 \ll x_3 - x_2$, $x_2 - x_1 \ll x_1$, approximated by using the asymptotic expansion (*A23*):

$$I_2 \approx K\left[(x_3^{2/3} - x_1^{2/3}) - \frac{3}{2B}\frac{(x_2 - x_1)^{4/3}}{x_1^{1/3}(x_3 - x_1)^{1/3}} + \ldots\right]. \tag{65}$$

By analogy, an approximate expression for I_3 can be found in the form

$$I_3 \approx K\left[(x_5^{2/3} - x_3^{2/3}) - \frac{3}{2B}\frac{(x_4 - x_3)^{4/3}}{x_3^{1/3}(x_5 - x_3)^{1/3}} + \ldots\right]. \tag{66}$$

Effect of Insertions on Diagnostic Sensitivity

The ratio of the partial currents to individual segments does not depend on the transport properties of the solution but it can depend sensitively on various kinematic features of the flow close to the wall, e.g. flow direction[4], nonlinear velocity profiles[12], or longitudinal changes of the shear rate[13]. The presence of insulating insertions reduces the differences between the partial currents and, as a result, diminishes the sensitivity of segmented probes to the forementioned flow features.

This loss of the diagnostic sensitivity is shown quantitatively in Fig. 2. for two strip segments of equal lengths L separated by insulating insertions of varying size G. The effect of insertion is negligible in the region $G/L < 0{\cdot}01$. An acceptable diagnostic sensitivity is maintained up to $G/L = 1$. The screening effect of the forward segment becomes negligible for $G/L > 100$. Analogous conclusions can be made for the system of three strip segments of equal size, as shown in Fig. 3.

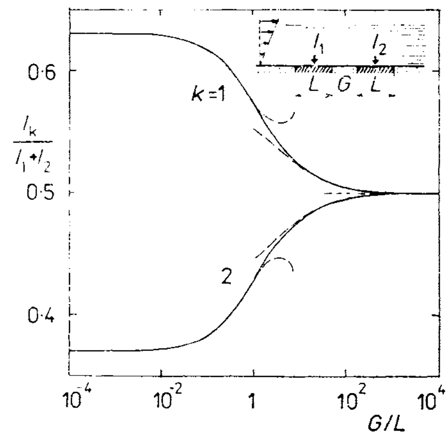

FIG. 2

Effect of insulating insertions on diagnostic sensitivity of a two-segment probe; dashed lines — asymptotes, Eqs (63)–(66)

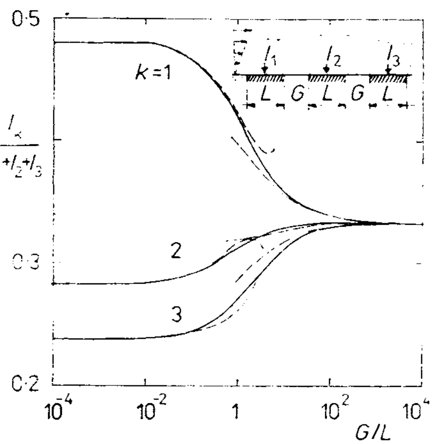

FIG. 3

Effect of insulating insertions on diagnostic sensitivity of a three-segment probe; dashed lines — asymptotes, Eqs (63)–(66)

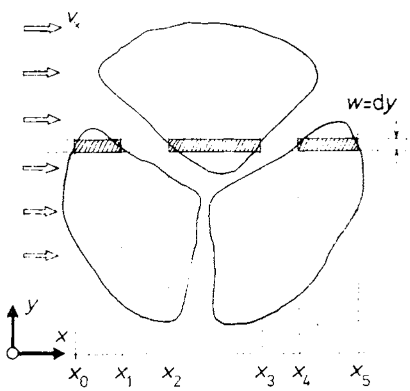

FIG. 4

Front view of an actual three-segment probe, with the scheme for the calculation of currents to the individual segments

Segmented Circular Probes

A typical three-segment probe for direction-sensitive shear rate measurements is shown in Fig. 4. The currents to the individual segments of such a probe can be calculated by dividing the area of the probe into a system of parallel differential strips. It can be seen from Fig. 4. that, in some cases, it is necessary to calculate the currents to three consecutive segments separated by insertions of considerable sizes. In the present paper, we have found convenient computation formulae, Eqs (*58*)–(*60*), for accomplishing this task. The direction characteristics of such non-ideal segmented probes will be analysed in our later report.

CONCLUSIONS

The equations of convective diffusion in the diffusion-layer approximation have been solved for the flow with constant velocity gradient q and longitudinally varying conditions at the wall which represent an alternation of electrochemically active segments and insulating insertions. The problem of two consecutive electrodes, known from the literature[6], has been solved here by using substantially simpler means. Explicit formulae have also been given for the system of three consecutive electrodes, which is of some importance in the theory of direction-sensitive electrodiffusion probes[4,13]. The main result consists in the reduction of the integral (*51*) to a single quadrature (*55*).

As shown by several authors[7,9], the suggested way of including the effect of insulating insertions on the current distribution is not limited to unidirectional kinematics of flow. The same formulae (*58*)–(*60*) can be used for any laminar flow, if the longitudinal coordinates x_k are replaced by the corresponding variables in accordance with Lighthill transformation.

The effect of former segments on the current to a segment placed down the stream is represented by a correction coefficient which depends only on geometrical simplexes of the electrode system. In particular, it is independent of the flow velocity and of the liquid properties. Such conclusions are, however, limited to the sizes of insertions and active segments which are substantially larger than the diffusion thickness[14] and, on the other hand, substantially smaller than the macroscopic scales of the flow. Fortunately, such conditions are fulfilled safely for the probes used in the electrodiffusion flow diagnostics.

APPENDIX

Beta Function and Related Integrals

The function $\mathrm{F}(x)$ is identical with the standard incomplete Beta function:

$$\mathrm{F}(x) \equiv B^{-1} \int_0^x (1 - s)^{-2/3} s^{-1/3} \, \mathrm{d}s \,, \qquad (A1)$$

$$B = \int_0^1 (1-s)^{-2/3} s^{-1/3} \, ds = \Gamma(\tfrac{1}{3})\,\Gamma(\tfrac{2}{3}) = 2\pi/\sqrt{3}\,, \tag{A2}$$

and it can also be expressed by elementary functions:

$$F(x) = \frac{1}{4} + \frac{3}{2\pi} \arctan\left[\frac{2}{\sqrt{3}}\left(\frac{x}{1-x}\right)^{1/3} - \frac{1}{\sqrt{3}}\right] - \frac{3}{2B} \ln\left[(1-x)^{1/3} + x^{1/3}\right] \tag{A3}$$

The following series expansions

$$F(x) \approx \frac{3}{2B} x^{2/3} \left[1 + \tfrac{2}{15} x + \ldots\right] \tag{A4}$$

with $x \to 0$, and

$$F(x) \approx 1 - \frac{3}{B}(1-x)^{1/3}\left[1 + \frac{1-x}{12} + \ldots\right] \tag{A5}$$

with $(1-x) \to 0$, are used in the main text for developing various asymptotic representations.

In some cases, the integrals of rational functions with fractional exponents can be expressed by elementary functions:

$$\frac{1}{3}\int_b^x \frac{ds}{(s-a)^{4/3}(x-s)^{2/3}} = \frac{1}{x-a}\left(\frac{x-b}{b-a}\right)^{1/3}. \tag{A6}$$

However, special functions must often be used:

$$\frac{1}{B}\int_a^b \frac{ds}{(s-a)^{1/3}(x-s)^{2/3}} = F\left(\frac{b-a}{x-a}\right), \tag{A7}$$

$$\frac{1}{B}\int_b^x \frac{ds}{(s-a)^{2/3}(x-s)^{1/3}} = F\left(\frac{x-b}{x-a}\right), \tag{A8}$$

$$\frac{1}{B}\int_0^b \frac{dt}{t^{1/3}(x+t)} = x^{-1/3}\, F\left(\frac{b}{x+b}\right), \tag{A9}$$

$$\frac{1}{B}\int_b^\infty \frac{dt}{t^{2/3}(x+t)} = x^{-2/3}\, F\left(\frac{x}{x+b}\right), \tag{A10}$$

$$\frac{1}{B}\int_b^a \left(\frac{s-a}{x-s}\right)^{2/3} \frac{ds}{s-v} = -\left(\frac{a-v}{x-v}\right)^{2/3} F\left(\frac{b-a}{b-v}\,\frac{x-v}{x-a}\right) + F\left(\frac{b-a}{x-a}\right), \tag{A11}$$

$$\frac{1}{B}\int_b^x \left(\frac{x-s}{s-a}\right)^{2/3} \frac{ds}{s-v} = \left(\frac{x-v}{a-v}\right)^{2/3} F\left(\frac{x-b}{b-v}\frac{a-v}{x-a}\right) - F\left(\frac{x-b}{x-a}\right), \quad (A12)$$

$$\frac{1}{B}\int_b^x \left(\frac{s-a}{x-s}\right)^{1/3} \frac{ds}{s-v} = -\left(\frac{a-v}{x-v}\right)^{1/3} F\left(\frac{x-b}{b-v}\frac{a-v}{x-a}\right) + F\left(\frac{x-b}{x-a}\right), \quad (A13)$$

$$\frac{1}{B}\int_a^b \left(\frac{x-s}{s-a}\right)^{1/3} \frac{ds}{s-v} = \left(\frac{x-v}{a-v}\right)^{1/3} F\left(\frac{b-a}{b-v}\frac{x-v}{x-a}\right) - F\left(\frac{b-a}{x-a}\right), \quad (A14)$$

$$\frac{1}{B}\int_b^x \frac{x^{1/3}a^{2/3}\,ds}{(x-s)^{1/3}(s-a)^{2/3}\,s} = F\left(\frac{a}{b}\frac{x-b}{x-a}\right), \quad (A15)$$

$$\frac{1}{B}\int_a^b \frac{x^{2/3}a^{1/3}\,ds}{(x-s)^{2/3}(s-a)^{1/3}\,s} = F\left(\frac{x}{b}\frac{b-a}{x-a}\right), \quad (A16)$$

$$\frac{1}{3}\int_a^b \frac{[1-F(a/t)]\,dt}{(x-t)^{4/3}} = (x-b)^{-1/3}\left[1-F\left(\frac{a}{b}\right)\right] - x^{-1/3}\left[1-F\left(\frac{a}{b}\frac{x-b}{x-a}\right)\right], \quad (A17)$$

$$\frac{2}{3}\int_a^b \frac{[1-F(a/t)]\,dt}{(x-t)^{1/3}} = x^{2/3} - a^{2/3} - x^{2/3}\,\Phi\left(\frac{a}{x},\frac{b}{x}\right). \quad (A18)$$

The auxiliary function Φ,

$$\Phi(\alpha,\beta) \equiv (1-\beta)^{2/3}\left[1-F\left(\frac{\alpha}{\beta}\right)\right] + F\left(\frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha}\right) - \alpha^{2/3}\,F\left(\frac{1-\beta}{1-\alpha}\right), \quad (A19)$$

has the following asymptotical representations – for $\alpha \to 0$:

$$\Phi(\alpha,\beta) \approx (1-\beta)^{2/3} - F(1-\beta)\,\alpha^{2/3} + \frac{2}{B}((1-\beta)/\beta)^{2/3}\,\alpha^{5/3}, \quad (A20)$$

for $(1-\alpha/\beta) \equiv s \to 0$:

$$\Phi(\alpha,\beta) \approx 1 - \beta^{2/3} + \tfrac{2}{3}\beta^{2/3}s - \frac{3\beta}{2B}(1-\beta)^{-1/3}\,s^{4/3} \approx$$

$$\approx 1 - \alpha^{2/3} - \frac{3\beta}{2B}(1-\alpha)^{-1/3}\,s^{4/3}, \quad (A21)$$

and for $(1-\beta) \equiv s \to 0$:

$$\Phi(\alpha,\beta) \approx [1-F(\alpha)]\,s^{2/3} - \left(1-\frac{1}{B}\right)\left(\frac{\alpha}{1-\alpha}\right)^{2/3} s^{5/3}. \quad (A22)$$

SYMBOLS

$B = 2\pi/\sqrt{3}$
c concentration field of depolarizer
c_0 bulk concentration of depolarizer
C^* normalized concentration field, Eq. (*9*)
$C(X)$ normalized surface concentration, Eq. (*11*)
$C_k(X)$ representation of $C(X)$ for k-th insertion
D diffusivity of depolarizer
d_x, d_z partial derivatives
$F(x)$ incomplete Beta function, Eq. (*A1*)
G lenght of an insulating insertion
I current, Eq. (*7*)
I_k partial current to k-th segment, Eq. (*8*)
J current density, Eq. (*6*)
K transport coefficient, Eq. (*61*)
L length of a segment
nF electric charge corresponding to the conversion of 1 mole of depolarizer
$N(X)$ normalized current, Eq. (*12*)
$\dot{N}(X) \equiv dN/dX$ normalized current density
q shear rate, Eq. (*1*)
v_x, v_y, v_z velocity components
x longitudinal coordinate (parallel to the flow)
x_k value of x for the boundary between segments and insertions, see Fig. 1.
$X = x/\mu$, $X_k = x_k/\mu$,
z normal coordinate (perpendicular to the wall)
$Z = z/\mu$
w width of a segment (size across the flow)
$\varkappa_k = x_k/x_5 = X_k/X_5$
Φ auxiliary function, Eq. (*A19*)
μ length scale, Eq. (*10*)

REFERENCES

1. Py B.: Int. J. Heat Mass Transfer *16*, 129 (1973).
2. Wichterle K., Žák L.: Czech. AO 231308 (1984).
3. Sobolík V., Mitschka P., Menzel T.: Czech. AO 262 823 (1989).
4. Wein O., Sobolík V.: Collect. Czech. Chem. Commun. *52*, 2169 (1987).
5. Sobolík V., Wein O., Gil O., Tribollet B.: Experiments in Fluids in press (1990).
6. Albery W. J., Bruckenstein S.: Trans. Faraday Soc. *62*, 1920 (1966).
7. Matsuda H.: J. Electroanal. Chem. *16*, 153 (1968).
8. Filinovskii V. Yu., Kadiya I. V., Nikolic B. Zh., Nakic M. V.: J. Electroanal. Chem. *54*, 39 (1974).
9. Suwono A., Daguenet M., Bodiot D.: Int. J. Heat Mass Transfer *19*, 239 (1976).
10. Filinovskii V. Yu.: Electrochim. Acta *25*, 309 (1980).
11. Carslaw H. S., Jaeger J. C.: *Conduction of Heat in Solids*. Clarendon Press, Oxford 1962.
12. Pokryvaylo N. A., Wein O., Kovalevskaya N. D.: *Electrodiffusion Diagnostics of Flow in Suspensions and Polymer Solutions* (in Russian). Nauka i Tekhnika, Minsk 1988.

13. Wein O., Sobolík V.: Collect. Czech. Chem. Commun. *54*, (1989).
14. Wein O., Kovalevskaya N. D.: Collect. Czech. Chem. Commun. *52*, 634 (1987).

Translated by the author (O.W.).
Translation revised by K. Micka.